

Linearization in ultrametric dynamics in fields of characteristic zero – equal characteristic case*

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Abstract

Let K be a complete ultrametric field of characteristic zero whose corresponding residue field \mathbb{k} is also of characteristic zero. We give lower and upper bounds for the size of linearization disks for power series over K near an indifferent fixed point. These estimates are maximal in the sense that there exist examples where these estimates give the exact size of the corresponding linearization disc. Similar estimates in the remaining cases, i.e. the cases in which K is either a p -adic field or a field of prime characteristic, were obtained in various papers on the p -adic case [5, 18, 35, 42] later generalized in [28], and in [29, 31] concerning the prime characteristic case.

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1 Introduction

A central issue in the study of dynamical systems is the local dynamics near periodic points. In particular, it is of great importance to know whether or not the dynamics is locally linearizable near a given periodic point. Recall that a power series f , over a complete valued field K , of the form

$$f(x) = \lambda x + a_2 x^2 + a_3 x^3 \dots, \quad \text{with } |\lambda| = 1, \text{ but not a root of unity,} \quad (1)$$

is said to be analytically linearizable at the indifferent fixed point at the origin if there is a convergent power series solution g to the following form

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of the Schröder functional equation

$$g \circ f(x) = \lambda g(x), \quad \lambda = f'(0), \quad (2)$$

which conjugates f to its linear part. The coefficients of the formal solution g of (2) must satisfy a recurrence relation of the form

$$b_k = \frac{1}{\lambda(1 - \lambda^{k-1})} C_k(b_1, \dots, b_{k-1}).$$

Intuitively, if λ is close to a root of unity we might run into a problem of small divisors as in the well-known complex case [4, 11, 33]. In 1942 Siegel proved in his celebrated paper [39] that the condition

$$|1 - \lambda^n| \geq Cn^{-\beta} \quad \text{for some real numbers } C, \beta > 0, \quad (3)$$

on λ is sufficient for convergence in the complex field case. Later, Brjuno [10] proved that the weaker condition

$$-\sum_{k=0}^{\infty} 2^{-k} \log \left(\inf_{1 \leq n \leq 2^{k+1}-1} |1 - \lambda^n| \right) < +\infty, \quad (4)$$

is sufficient. In fact, for quadratic polynomials, the Brjuno condition is not only sufficient but also necessary as shown by Yoccoz [44].

Since then, there has been an increasing interest in the ultrametric analogue of complex dynamics, see e.g. [1–3, 6–8, 12, 17–24, 26, 27, 29, 30, 32, 34, 36, 37, 40, 41].

It is known since a work of Herman and Yoccoz [15] that Siegel’s linearization theorem is true also in the ultrametric case. Moreover, in the one-dimensional case, for fields of characteristic zero the Siegel condition is always satisfied. In the two-dimensional p -adic case, the conjugacy may diverge as shown in [15]. Recently, the multi-dimensional p -adic case has been studied in more detail by Viegue in his thesis [43].

However, as noted by Herman and Yoccoz, in fields of prime characteristic there is a problem of small divisors also in the one-dimensional case; in general the multiplier does not satisfy the Siegel nor the weaker Brjuno condition. One might therefore conjecture, as Herman [14], that *for a locally compact, complete valued field of prime characteristics, the formal conjugacy ‘usually’ diverges, even for polynomials of one variable*. Indeed, as shown in the papers [29, 31] like in complex dynamics, the formal solution may diverge also in the one-dimensional case. On the other hand, in [29, 31] it was also proven that the conjugacy may still converge due to considerable cancellation of small divisor terms; the same multiplier λ may yield convergence for some f but not for others. This brings about a problem of a combinatorial nature of seemingly great complexity and a complete description is yet to be found. For example, we have the following open problem stated in [29].

Open Problem *Let K be of characteristic $p > 0$. Is there a polynomial of the form $f(x) = \lambda x + O(x^2) \in K[x]$, with λ not a root of unity satisfying $|1 - \lambda^n| < 1$ for some $n \geq 1$, and containing a monomial of degree prime to p , such that the formal conjugacy g converges?*

For a more thorough treatment of the problem and its relation to the complex case, the reader can consult [29].

In case of convergence, one can estimate the radius of convergence for the corresponding *linearization disc*¹ Δ_f , i.e. the maximal disc U , about the origin, such that the full conjugacy $g \circ f \circ g^{-1}(x) = \lambda x$, holds for all $x \in U$. Estimates of linearization discs have appeared in several papers concerning the p -adic case [5, 18, 35, 42] later generalized in [28], and in [29, 31] concerning the prime characteristic case.

In this paper we consider the remaining case, namely the case in which both K and the associated residue field \mathbb{k} are of characteristic zero. For example, K could be the function field $\mathbb{C}((T))$; the field of formal Laurent series in variable T over the complex numbers. Our main result can be stated in the following way.

Theorem 1.1. *Let $\text{char } K = \text{char } \mathbb{k} = 0$ and $f \in K[[x]]$ be of the form (1) with $a = \sup_{i \geq 2} |a_i|^{1/(i-1)}$. Let $\bar{\lambda}$ be the representative of λ in \mathbb{k} and $\Gamma(\mathbb{k})$ be the set of roots of unity in \mathbb{k} . Then, the corresponding linearization disc Δ_f can be estimated as follows.*

1. *If $\bar{\lambda} \notin \Gamma(\mathbb{k})$, then $D_{1/a}(0) \subseteq \Delta_f \subseteq \overline{D}_{1/a}(0)$. If in addition $a = \max_{i \geq 2} |a_i|^{1/(i-1)}$ is attained as for polynomials, or f diverges on the sphere $S_{1/a}(0)$, then $\Delta_f = D_{1/a}(0)$.*
2. *If $\bar{\lambda} \in \Gamma(\mathbb{k})$ and m is the smallest integer such that $|1 - \lambda^m| < 1$. Then, $D_\rho(0) \subseteq \Delta_f \subseteq \overline{D}_{1/a}(0)$, where $\rho = \sqrt[m]{|1 - \lambda^m|}/a$. If $a = \max_{i \geq 2} |a_i|^{1/(i-1)}$ or f diverges on the sphere $S_{1/a}(0)$, then $D_\rho(0) \subseteq \Delta_f \subseteq D_{1/a}(0)$.*

These estimates are maximal in the sense that there exist examples of such f which have a periodic point on the sphere $S_\rho(0)$, breaking the conjugacy there.

These estimates take a simpler form than the corresponding p -adic case [28]. Second, the radius of the linearization disc is in general larger than in the p -adic case. These two facts both stem from the fact that the geometry of the roots of unity in \mathbb{C}_p is more complex than in the equal characteristic case.

¹Here we use the term ‘linearization disc’ rather than ‘Siegel disc’, because in ultrametric dynamics the Siegel disc is often referred to as the larger maximal disc on which f is one-to-one.

2 Preliminaries

Throughout this paper K is a ultrametric field of characteristic zero ($\text{char } K = 0$), complete with respect to a nontrivial absolute value $|\cdot|$. That is, $|\cdot|$ is a multiplicative function from K to the nonnegative real numbers with $|x| = 0$ precisely when $x = 0$, satisfying the following strong or ultrametric triangle inequality:

$$|x + y| \leq \max[|x|, |y|], \quad \text{for all } x, y \in K, \quad (5)$$

and nontrivial in the sense that it is not identically 1 on K^* , the set of all nonzero elements in K . One useful consequence of ultrametricity is that for any $x, y \in K$ with $|x| \neq |y|$, the inequality (5) becomes an equality. In other words, if $x, y \in K$ with $|x| < |y|$, then $|x + y| = |y|$.

In this context it is standard to denote by \mathcal{O} , the ring of integers of K , given by $\mathcal{O} = \{x \in K : |x| \leq 1\}$, by \mathcal{M} the unique maximal ideal of \mathcal{O} , given by $\mathcal{M} = \{x \in K : |x| < 1\}$, and by k the corresponding *residue field*

$$\mathbb{k} = \mathcal{O}/\mathcal{M}.$$

Note that if $x, y \in \mathcal{O}$ reduce to *residue classes* $\bar{x}, \bar{y} \in k$, then $|x - y|$ is 1 if $\bar{x} \neq \bar{y}$, and it is strictly less than 1 otherwise. Note also that if K has positive characteristic p , then also $\text{char } \mathbb{k} = p$; but if $\text{char } K = 0$, then \mathbb{k} could have characteristic p (the p -adic case) or 0. In this paper we mainly consider the latter, equal charactersitic case $\text{char } K = \text{char } \mathbb{k} = 0$.

Example 2.1 ($\text{char } K = \text{char } \mathbb{k} = 0$). *Let F be a field of characteristic zero, e.g. F could be either \mathbb{Q} , \mathbb{Q}_p , \mathbb{R} or \mathbb{C} . Let $F((T))$ be the field of all formal Laurent series in variable T , with coefficients in the field F . An element $x \in K$ is of the form*

$$x = \sum_{i \geq i_0} x_i T^i, \quad x_{i_0} \neq 0, \quad x_i \in F, \quad (6)$$

for some integer $i_0 \in \mathbb{Z}$. Given $0 < \epsilon < 1$, we define an absolute value $|\cdot|$ on K such that $|T| = \epsilon$ and

$$\left| \sum_{i \geq i_0} x_i T^i \right| = \epsilon^{i_0}. \quad (7)$$

Hence, $\pi = T$ is a uniformizer of K . Furthermore, K is complete with respect to $|\cdot|$ and, analogously to the p -adic numbers, can be viewed as the completion of the field of rational functions $F(T)$ over F with respect to the absolute value defined by (7). Note that in this case the residue field $\mathbb{k} = F$.

Note also that i_0 is the order of the zero (or if negative, the order of the pole) of x at $T = 0$. Moreover, $|\cdot|$ is the trivial absolute value on F ,

the subfield of K consisting of all constant series in K . As for the p -adic numbers, we can construct a completion \widehat{K} of an algebraic closure of K with respect to an extension of $|\cdot|$. Then \widehat{K} is a complete, algebraically closed ultrametric field, and its residue field $\widehat{\mathbb{k}}$ is an algebraic closure of $\mathbb{k} = F$. It follows that $\widehat{\mathbb{k}}$ has to be infinite. The value group $|\widehat{K}^*|$, that is the set of real numbers which are actually absolute values of non-zero elements of \widehat{K} , will consist of all rational powers of ϵ , rather than just integer powers of ϵ as in $|K^*|$. In particular, the absolute value is discrete on K but not on \widehat{K} .

We use the following notation for discs. Given an element $x \in K$ and real number $r > 0$ we denote by $D_r(x)$ the open disc of radius r about x , by $\overline{D}_r(x)$ the closed disc, and by $S_r(x)$ the sphere of radius r about x . To omit confusion, we sometimes write $D_r(x, K)$ rather than $D_r(x)$ to emphasize that the disc is considered as a disc in K .

If $r \in |K^*|$ (that is if r is actually the absolute value of some nonzero element of K), we say that $D_r(x)$ and $\overline{D}_r(x)$ are *rational*. Note that $S_r(x)$ is non-empty if and only if $\overline{D}_r(x)$ is rational. If $r \notin |K^*|$, then we will call $D_r(x) = \overline{D}_r(x)$ an *irrational* disc. In particular, if $a \in K \subset \mathbb{C}_p$ and $r = |a|^s$ for some rational number $s \in \mathbb{Q}$, then $D_r(x)$ and $\overline{D}_r(x)$ are rational considered as discs in the algebraic closure \mathbb{C}_p . However, they may be irrational considered as discs in K . Note that all discs are both open and closed as topological sets, because of ultrametricity. However, as we will see in Section 2.1 below, power series distinguish between rational open, rational closed, and irrational discs.

2.1 Mapping properties

Let f be a power series over K of the form

$$f(x) = \sum_{i=0}^{\infty} a_i(x - \alpha)^i, \quad a_i \in K.$$

Then, f converges on the open disc $D_{R_f}(\alpha)$ of radius

$$R_f = \frac{1}{\limsup |a_i|^{1/i}}, \quad (8)$$

and diverges outside the closed disc $\overline{D}_{R_f}(\alpha)$ in K . The power series f converges on the sphere $S_{R_f}(\alpha)$ if and only if

$$\lim_{i \rightarrow \infty} |a_i| R_f^i = 0.$$

The basic mapping properties of ultrametric power series on discs are given by the following generalization by Benedetto [7], of the Weierstrass Preparation Theorem [9, 13, 25].

Proposition 2.1 (Lemma 2.2 [7]). *Let K be algebraically closed. Let $f(x) = \sum_{i=0}^{\infty} a_i(x - \alpha)^i$ be a nonzero power series over K which converges on a rational closed disc $U = \overline{D}_R(\alpha)$, and let $0 < r \leq R$. Let $V = \overline{D}_r(\alpha)$ and $V' = D_r(\alpha)$. Then*

$$\begin{aligned} s &= \max\{|a_i|r^i : i \geq 0\}, \\ d &= \max\{i \geq 0 : |a_i|r^i = s\}, \quad \text{and} \\ d' &= \min\{i \geq 0 : |a_i|r^i = s\} \end{aligned}$$

are all attained and finite. Furthermore,

- a. $s \geq |f'(x_0)| \cdot r$.
- b. *if $0 \in f(V)$, then f maps V onto $\overline{D}_s(0)$ exactly d -to-1 (counting multiplicity).*
- c. *if $0 \in f(V')$, then f maps V' onto $D_s(0)$ exactly d' -to-1 (counting multiplicity).*

We will consider the case $a_0 = 0$ in more detail. For our purpose, it is then often more convenient to state Proposition 2.1 in the following way.

Proposition 2.2. *Let K be algebraically closed and let $h(x) = \sum_{i=1}^{\infty} c_i(x - \alpha)^i$ be a power series over K .*

- 1. *Suppose that h converges on the rational closed disc $\overline{D}_R(\alpha)$. Let $0 < r \leq R$ and suppose that*

$$|c_i|r^i \leq |c_1|r \quad \text{for all } i \geq 2. \tag{9}$$

Then, h maps the open disc $D_r(\alpha)$ one-to-one onto $D_{|c_1|r}(0)$. Furthermore, if

$$d = \max\{i \geq 1 : |c_i|r^i = |c_1|r\},$$

then h maps the closed disc $\overline{D}_r(\alpha)$ onto $\overline{D}_{|c_1|r}(0)$ exactly d -to-1 (counting multiplicity).

- 2. *Suppose that h converges on the rational open disc $D_R(\alpha)$ (but not necessarily on the sphere $S_R(0)$). Let $0 < r \leq R$ and suppose that*

$$|c_i|r^i \leq |c_1|r \quad \text{for all } i \geq 2.$$

Then, h maps $D_r(\alpha)$ one-to-one onto $D_{|c_1|r}(0)$.

Now, suppose that f has a fixed point at x_0 (so that $f(x_0) = x_0$) and that $|f'(x_0)| = 1$. As a consequence of Proposition 2.1, f is not only one-to-one but a bijective isometry on some non-empty disc about x_0 . The maximal such disc is given by the following proposition.

Proposition 2.3. *Let K be algebraically closed. Let $f \in K[[x]]$ be convergent on some non-empty disc about $x_0 \in K$. Suppose that $f(x_0) = x_0$ and $|f'(x_0)| = 1$, and write*

$$f(x) = x_0 + \lambda(x - x_0) + \sum_{i \geq 2} a_i(x - x_0)^i, \quad a = \sup_{i \geq 2} |a_i|^{1/(i-1)}.$$

Let M be the largest disc, with $x_0 \in M$, such that $f : M \rightarrow M$ is bijective (and hence isometric). Then $M = D_{1/a}(x_0)$ if either $\max_{i \geq 2} |a_i|^{1/(i-1)}$ is attained (as for polynomials) or f diverges on $S_{1/a}(x_0)$. Otherwise, $M = \overline{D}_{1/a}(x_0)$.

A proof is given in [28]. It follows that if f converges on the sphere $S_{1/a}(x_0)$ but fails to be one-to-one there, then there is a point $x \in S_{1/a}(x_0)$ such that $f(x) = x_0 = f(x_0)$. This is always the case when f is a polynomial.

That f may diverge on $S_{1/a}(x_0)$ follows since, for example, the power series $f(x) = \lambda x + \sum_{i=2}^{\infty} (a_2)^{i-1} x^i$ converges if and only if $|x| < 1/|a_2| = 1/a$.

Furthermore, for every $x \in M$, $|f(x) - x_0| = |x - x_0|$ and hence all spheres in M are invariant under f .

Remark 2.1. *Recall that the discs $D_{1/a}(0)$ and $\overline{D}_{1/a}(0)$ are rational if and only if $a = \sup_{i \geq 2} |a_i|^{1/(i-1)} \in |K|$. If the maximum $a = \max_{i \geq 2} |a_i|^{1/(i-1)}$ exists, and K is algebraically closed, then $a \in |K|$. This is always the case if f is a polynomial. If f is not a polynomial and the maximum fails to exist we may have $\sup_{i \geq 2} |a_i|^{1/(i-1)} \notin |K|$. Let $K = \mathbb{C}_p$. Let β be an irrational number and let p_n/q_n be the n -th convergent of the continued fraction expansion of β . Let the sequence $\{a_i \in \mathbb{Q}_p\}_{i \geq 2}$ satisfy*

$$|a_i| = \begin{cases} p^{p_n}, & \text{if } i-1 = q_n \text{ and } p_n/q_n < \beta, \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\sup_{i \geq 2} |a_i|^{1/(i-1)} = p^\beta \notin |K| = \{p^r : r \in \mathbb{Q}\} \cup \{0\}.$$

For more information on ultrametric power series the reader can consult [38]. From a dynamical point of view, the paper [7] contains many useful results on ultrametric analogues of complex analytic mapping theorems relevant for dynamics.

2.2 The linearization disc

The results above have some important implications for linearization discs. We use the following definition of a linearization disc. Let K be a complete ultrametric field of characteristic zero. Suppose that $f \in K[[x]]$ has an indifferent fixed point $x_0 \in K$, with multiplier $\lambda = f'(x_0)$, not a root of unity.

By [16], there is a unique formal power series solution g , with $g(x_0) = 0$ and $g'(x_0) = 1$, to the following form of the Schröder functional equation

$$g \circ f(x) = \lambda g(x).$$

If the formal solution g converges on some non-empty disc about x_0 , then the corresponding *linearization disc* of f about x_0 , denoted by $\Delta_f(x_0)$, is defined as the largest disc $U \subset K$, with $x_0 \in U$, such that the Schröder functional equation holds for all $x \in U$, and g converges and is one-to-one on U . We will often refer to g as the *conjugacy function*.

This notion of a linearization disc is well-defined since, by proposition 2.3, there always exist a largest disc on which g is one-to-one (provided that g is convergent). Recall that by the ultrametric Siegel theorem by Herman and Yoccoz [16], the formal solution g always converges if $\text{char } K = 0$.

As a consequence of the results stated in the section above, both f and the conjugacy g turn out to be one-to-one and isometric on a ultrametric linearization disc.

Proposition 2.4. *Let K be algebraically closed. Suppose that $f \in K[[x]]$ has a linearization disc $\Delta_f(x_0)$ about $x_0 \in K$. Let g , with $g(x_0) = 0$ and $g'(x_0) = 1$, be the corresponding conjugacy function. Then, both $g : \Delta_f(x_0) \rightarrow g(\Delta_f(x_0))$ and $f : \Delta_f(x_0) \rightarrow \Delta_f(x_0)$ are bijective and isometric. In particular, if $x_0 = 0$, then $g(\Delta_f(x_0)) = \Delta_f(x_0)$. Furthermore, $\Delta_f(x_0) \subseteq M \subseteq \overline{D}_{1/a}(x_0)$, where M and a are defined as in Lemma 2.3.*

Hence, the radius of a linearization disc $\Delta_f(x_0)$ is equal to to that of $g(\Delta_f(x_0))$. In particular, the radius of a linearization disc is independent of the location of the fixed point x_0 . Therefore, we shall, without loss of generality, henceforth assume that $x_0 = 0$.

Remark 2.2. *All the results in this and the previous section, except for Proposition 2.1, hold also in the case that K is not algebraically closed, with the modification that the mappings are one-to-one but not necessarily surjective.*

3 Proof of the main theorem

From now on $\text{char } K = \text{char } \mathbb{k} = 0$. As noted in the previous section, we may, without loss of generality, assume that f has its fixed point at the origin, and that $f \in \mathcal{F}_{\lambda,a}$, as defined below. Let $\lambda \in K$ be such that

$$|\lambda| = 1, \quad \text{but } \lambda^n \neq 1, \quad \forall n \geq 1, \quad (10)$$

and let a be a real number. We shall associate with the pair (λ, a) a family $\mathcal{F}_{\lambda,a}$ of power series defined by

$$\mathcal{F}_{\lambda,a} := \left\{ \lambda x + \sum a_i x^i \in \mathbb{C}_p[[x]] : a = \sup_{i \geq 2} |a_i|^{1/(i-1)} \right\}. \quad (11)$$

It follows that each $f \in \mathcal{F}_{\lambda,a}$ is convergent on $D_{1/a}(0)$, and by Proposition 2.3 $f : D_{1/a}(0) \rightarrow D_{1/a}(0)$ is bijective and isometric.

As K is of characteristic zero, we may, by the ultrametric Siegel theorem [16], associate with f a unique convergent power series solution g to the Schröder functional equation, of the form

$$g(x) = x + \sum_{k \geq 2} b_k x^k,$$

and a corresponding linearization disc about the origin

$$\Delta_f := \Delta_f(0).$$

Recall that by Proposition 2.4, since $x_0 = 0$, the linearization disc Δ_f is the largest disc $U \subset K$ about the origin such that the full conjugacy $g \circ f \circ g^{-1}(x) = \lambda x$ holds for all $x \in U$.

Given $f \in \mathcal{F}_{\lambda,a}$, Proposition 2.4 yields the following concerning Δ_f .

Lemma 3.1. *Let $f \in \mathcal{F}_{\lambda,a}$. Then f has a linearization disc Δ_f about the origin in K . Let g , with $g(0) = 0$ and $g'(0) = 1$, be the corresponding conjugacy function. Then, the following two statements hold:*

- 1) *Both $g : \Delta_f \rightarrow \Delta_f$ and $f : \Delta_f \rightarrow \Delta_f$ are bijective and isometric.*
- 2) *$\Delta_f \subseteq \overline{D}_{1/a}(0)$. If $a = \max_{i \geq 2} |a_i|^{1/(i-1)}$ or f diverges on the sphere $S_{1/a}(0)$, then $\Delta_f \subseteq D_{1/a}(0)$.*

Also note the following lemma concerning estimates of the coefficients of the conjugacy.

Lemma 3.2. *Let $f \in \mathcal{F}_{\lambda,a}$. Then, the coefficients of the conjugacy function g satisfy*

$$|b_k| \leq \left(\prod_{n=1}^{k-1} |1 - \lambda^n| \right)^{-1} a^{k-1}, \quad (12)$$

for all $k \geq 2$.

Proof. The coefficients of the conjugacy g must satisfy the recurrence relation

$$b_k = \frac{1}{\lambda(1 - \lambda^{k-1})} \sum_{l=1}^{k-1} b_l \left(\sum \frac{l!}{\alpha_1! \cdots \alpha_k!} a_1^{\alpha_1} \cdots a_k^{\alpha_k} \right) \quad (13)$$

where $\alpha_1, \alpha_2, \dots, \alpha_k$ are nonnegative integer solutions of

$$\begin{cases} \alpha_1 + \dots + \alpha_k = l, \\ \alpha_1 + 2\alpha_2 + \dots + k\alpha_k = k, \\ 1 \leq l \leq k-1. \end{cases} \quad (14)$$

Note that the factorial factors $l!/\alpha_1! \cdots \alpha_k!$ are always integers and thus of modulus less than or equal to 1. Also recall that $|a_i| \leq a^{i-1}$. It follows that

$$|b_k| \leq \left(\prod_{n=1}^{k-1} |1 - \lambda^n| \right)^{-1} a^\alpha,$$

for some integer α . In view of equation (14) we have

$$\sum_{i=2}^k (i-1)\alpha_i = k-l.$$

Consequently, since $|a_i| \leq a^{i-1}$, we obtain

$$\prod_{i=2}^k |a_i|^{\alpha_i} \leq \prod_{i=2}^k a^{(i-1)\alpha_i} = a^{k-l}. \quad (15)$$

Now we use induction over k . By definition $b_1 = 1$ and, according to the recursion formula (13), $|b_2| \leq |1 - \lambda|^{-1}|a_2| \leq |1 - \lambda|^{-1}|a|$. Suppose that

$$|b_l| \leq \left(\prod_{n=1}^{l-1} |1 - \lambda^n| \right)^{-1} a^{l-1}$$

for all $l < k$. Then

$$|b_k| \leq \left(\prod_{n=1}^{k-1} |1 - \lambda^n| \right)^{-1} a^{l-1} \max \left\{ \prod_{i=2}^k |a_i|^{\alpha_i} \right\},$$

and the lemma follows by the estimate (15). \square

In the following we show how to calculate the distance $|1 - \lambda^n|$ for an arbitrary integer $n \geq 1$. Applying Proposition 2.2 to the estimate in the above lemma we can then estimate the disc on which the conjugacy function g is one-to-one.

Let $\lambda \in S_1(0)$, be an element in the unit sphere. We are concerned with calculating the distance

$$|1 - \lambda^n|, \quad \text{for } n = 1, 2, \dots$$

Recall that if $x, y \in \overline{D}_1(0)$, then $|x - y| < 1$ if and only if the reductions $\overline{x}, \overline{y}$ belong to the same residue class. Consequently,

$$|1 - \lambda^n| < 1 \iff \overline{\lambda}^n - 1 = 0 \quad \text{in } \mathbb{k}. \quad (16)$$

Hence, the behavior of $1 - \lambda^n$ falls into one of two categories, depending on whether the reduction of λ is a root of unity or not. For convenience, denote by $\Gamma(\mathbb{k})$ the set of roots of unity in \mathbb{k} . More precisely, we have the following lemma.

Lemma 3.3. *Let $\text{char } K = \text{char } \mathbb{k} = 0$. Suppose $\lambda \in S_1(0)$. Then*

1. $\bar{\lambda} \notin \Gamma(\mathbb{k}) \iff |1 - \lambda^n| = 1$ for all integers $n \geq 1$.
2. If $\bar{\lambda} \in \Gamma(\mathbb{k})$ then there is a smallest integer $m \geq 1$ such that $|1 - \lambda^m| < 1$. Moreover,

$$|1 - \lambda^n| = \begin{cases} 1, & \text{if } m \nmid n, \\ |1 - \lambda^m|, & \text{if } m \mid n. \end{cases} \quad (17)$$

Proof. First note that since the characteristic of the residue field is zero, $|\binom{l}{k}| = 1$ for all binomials $\binom{l}{k}$. By ultrametricity (5) we then have

$$|((\lambda^m - 1) + 1)^l - 1| = \left| \sum_{k=1}^l \binom{l}{k} (\lambda^m - 1)^k \right| = |\lambda^m - 1|,$$

as required. \square

Note that category 2 in Lemma 3.3 is always non-empty since $1, -1 \in \mathbb{Q} \subseteq \mathbb{k}$.

With these results at hand, we are now in a position to prove our main result.

Theorem 3.1. *Let $\text{char } K = \text{char } \mathbb{k} = 0$ and $f \in \mathcal{F}_{\lambda,a}$. Then, the corresponding linearization disc Δ_f can be estimated as follows.*

1. If $\bar{\lambda} \notin \Gamma(\mathbb{k})$, then $D_{1/a}(0) \subseteq \Delta_f \subseteq \overline{D}_{1/a}(0)$. If, in addition, $a = \max_{i \geq 2} |a_i|^{1/(i-1)}$ as for polynomials, or f diverges on the sphere $S_{1/a}(0)$, then $\Delta_f = D_{1/a}(0)$.
2. If $\bar{\lambda} \in \Gamma(\mathbb{k})$ and m is the smallest integer such that $|1 - \lambda^m| < 1$. Then, $D_\rho(0) \subseteq \Delta_f \subseteq \overline{D}_{1/a}(0)$, where $\rho = \sqrt[m]{|1 - \lambda^m|}/a$. If $a = \max_{i \geq 2} |a_i|^{1/(i-1)}$ or f diverges on the sphere $S_{1/a}(0)$, then $D_\rho(0) \subseteq \Delta_f \subseteq D_{1/a}(0)$.

These estimates are maximal in the sense that there exist examples of such f which have a periodic point on the sphere $S_\rho(0)$, breaking the conjugacy there.

Proof. The first statement follows immediately from Lemma 3.1, Lemma 3.2, and Lemma 3.3. To see that f may have a periodic point on the boundary breaking the conjugacy there we consider the polynomial $f(x) = \lambda x + a_n x^n$ for some integer $n \geq 2$. Note that we can choose $a = |a_n|^{1/(n-1)}$ in this case. Moreover $\hat{x} = [(1 - \lambda)/a_n]^{1/(n-1)}$ is fixed under f . But, since $|1 - \lambda| = 1$, we have $|\hat{x}| = 1/a$ so that \hat{x} sits on the sphere $S_{1/a}(0)$.

Now we consider the second statement. In this case, Lemma 3.3 implies that

$$|b_k| \leq \frac{1}{|1 - \lambda^m|^{[k-1/m]}} a^{k-1}.$$

It follows that the conjugacy g converges on the open disk of radius

$$\left(\limsup |b_k|^{1/k}\right)^{-1} \geq |1 - \lambda^m|^{\frac{1}{m}} a^{-1} = \rho.$$

Moreover,

$$|b_k| \leq |1 - \lambda^m|^{-\frac{k-1}{m}} a^{k-1} = \rho^{-(k-1)},$$

Consequently,

$$|b_k| \rho^k \leq \rho = |b_1| \rho,$$

In view of Proposition 2.2, $g : D_\rho(0) \rightarrow D_\rho(0)$ is bijective. Recall that by Lemma 2.3 $f : D_{1/a}(0) \rightarrow D_{1/a}(0)$ is a bijection. Moreover, $1/a > \rho$. Consequently, the linearization disc $\Delta_f \supseteq D_\rho(0)$ as required. This estimate of Δ_f is maximal in the sense that all f of the form $f(x) = \lambda x + a_2 x^2$ have a periodic fixed point on the sphere $S_\rho(0)$. \square

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